Point Functions :→ When a physical quantity takes different values at different points in space, it is called as a point function or field. If the point function is a scalar one, then it is called as a scalar field. If the physical quantity has a well defined magnitude as well as direction at different points of space, then it is called as a vector point function or vector field.

Examples of Scalar point function or Scalar field :–

Electric potential, temperature, pressure etc.

Examples of vector field :→

Electric field intensity, velocity field (in Streamlined flow), Magnetic field induction etc.

A scalar point function is said to be continuous in a region of space, if it suffers no abrupt change in it's magnitude as we displace infinitesimally from one point to an immediate neighbourhood point.

A vector field is said to be continuous in the defined region of space if it suffers no abrupt change in it's magnitude and direction (both) as we displace infinitesimally from one point to an immediate neighbourhood point in the region of space where the point function is well defined.

[A point function in a region of space (also called as domain) is said to be well defined if it has one and only one value at any point of that region].

Concept of Level Surfaces :→

Every well defined continuous scalar field is characterised by a set of parallel level surfaces.

"A level surface is one at every point of which the scalar point function takes the same value".

Characteristics of Level Surfaces :→

(i) The value of the scalar point function at all points of the level surface is same.

(ii) No two level surfaces ever intersect. If they will intersect, at the point of intersection, there will be two values for the scalar point function which is impossible.

(iii) The level surfaces are closely spaced at the region where the scalar point function changes faster with distance and they are far apart where the scalar point function changes slow.

Concept of Lines of Forces: →

Every continuous vector field is characterised by lines of forces.

"Lines of forces in a vector field are imaginary curves so drawn that at any point of this curve, if a tangent is drawn, it gives the direction of vector physical quantity at that point.

Characteristics of Lines of forces: →

(i) At every point of a lines of force if a tangent will be drawn, it will give the direction of vector field (i.e vector point function at that point). (Fig-1)

(ii) Two lines of forces never intersect. Because, if they will intersect, at the point of intersection, two tangents can be drawn showing two directions for vector field. This is impossible. (Fig-2)

(iii) The lines of forces are close at the region where the magnitude of vector field is more and far apart where the strength of vector field is weak. (Fig-3)

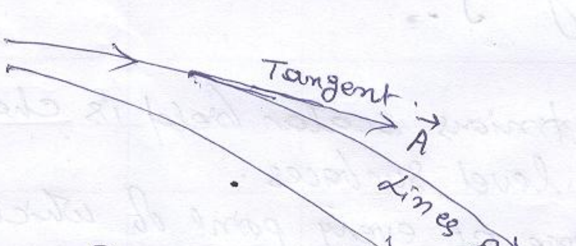


Fig-1

(Tangent to a Lines of force gives direction of vector point function)

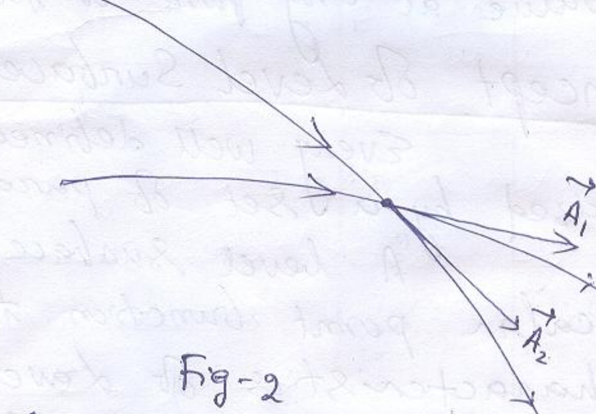


Fig-2

(If two lines of forces intersect, two tangents can be drawn at the point of intersection showing two directions for vector point function)

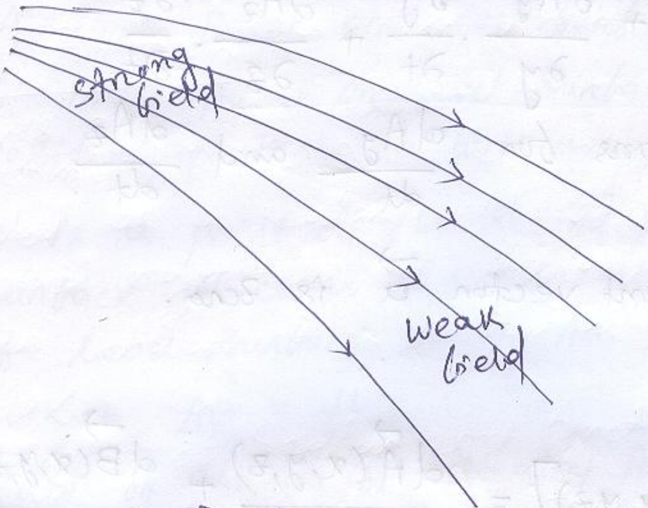


Fig-3

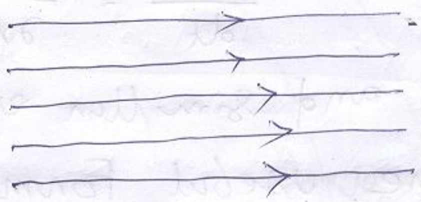
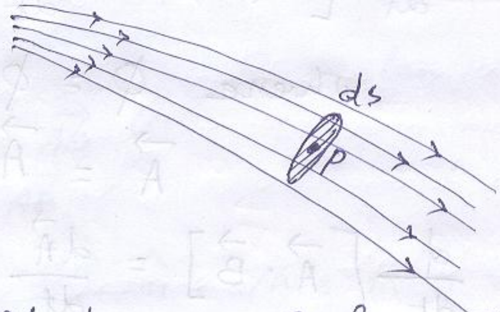


Fig-4
(pictorial representation of a uniform vector field where the lines of forces are parallel and equidistant from each other)

(iv) The no. of lines of forces passing through unit area drawn centred about a point and held perpendicular to the direction of lines of forces is proportional to the magnitude of vector field at that point.

$$|\vec{A}|_{at\ 'p'} \propto \frac{dN}{ds}$$



If $dN \rightarrow$ No. of lines of forces passing through area ds drawn centred about p and held perpendicular to lines of forces

Derivative of a Scalar point function: →

Since a Scalar point function is a function of space points, hence:-

In cartesian system, $\phi = \phi(x, y, z)$

$$\Rightarrow \frac{d\phi}{dt} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial t}$$

where 't' is a variable (independent)

Derivative of a vector point function in Cartesian System: →

A vector point function in Cartesian System is written as: →

$$\vec{A} = \vec{A}(x, y, z) = \hat{i} A_x(x,y,z) + \hat{j} A_y(x,y,z) + \hat{k} A_z(x,y,z)$$

$$\Rightarrow \frac{d\vec{A}}{dt} = \hat{i} \frac{dA_x(x,y,z)}{dt} + \hat{j} \frac{dA_y(x,y,z)}{dt} + \hat{k} \frac{dA_z(x,y,z)}{dt}$$

4

where $\dfrac{dA_x}{dt} = \dfrac{\partial A_x}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial A_y}{\partial y}\cdot\dfrac{\partial y}{\partial t} + \dfrac{\partial A_z}{\partial z}\cdot\dfrac{\partial z}{\partial t}$

and simillar expressions for $\dfrac{dA_y}{dt}$ and $\dfrac{dA_z}{dt}$.

Some useful Formulas :→

1 → Derivative of a constant vector $\vec{a}$ is zero.

$$\frac{d\vec{a}}{dt} = 0$$

2 → $$\frac{d}{dt}\left[\vec{A}(x,y,z) \pm \vec{B}(x,y,z)\right] = \frac{d\vec{A}(x,y,z)}{dt} \pm \frac{d\vec{B}(x,y,z)}{dt}$$

3 → $$\frac{d}{dt}\left[\phi\vec{A}\right] = \frac{d\phi}{dt}\vec{A} + \phi\cdot\frac{d\vec{A}}{dt}$$

where $\phi = \phi(x,y,z) \to$ a scalar point function.

$\vec{A} = \vec{A}(x,y,z) \to$ a vector point function.

4 → $$\frac{d}{dt}\left[\vec{A}\cdot\vec{B}\right] = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

5 → $$\frac{d}{dt}\left[\vec{A}\times\vec{B}\right] = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$ [Mark that order of cross product of $\vec{A}$ and $\vec{B}$ is not to be changed.

6 → $$\frac{d}{dt}\left[\vec{a}\,\phi\right] = \vec{a}\,\frac{d\phi}{dt}$$ [where $\vec{a}$ = constant].

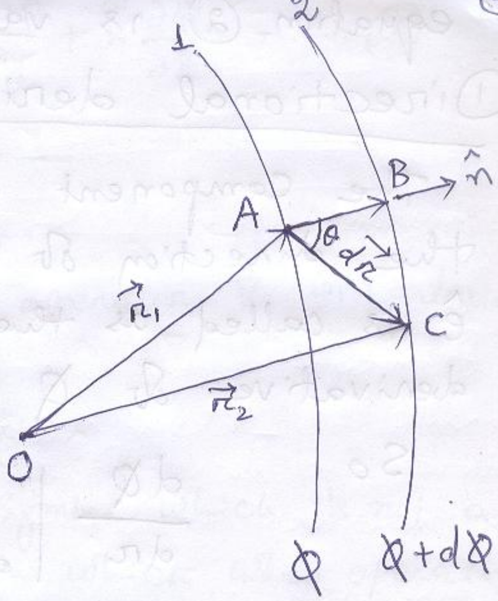
7 → $$\frac{d}{dt}\left[\vec{A}\cdot(\vec{B}\times\vec{C})\right] = \frac{d\vec{A}}{dt}\cdot(\vec{B}\times\vec{C}) + \vec{A}\cdot\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \vec{A}\cdot\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)$$

Note:→ In Cartesian co-ordinate system, with change of space point, the co-ordinates x, y and z changes but the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ remains unaffected. So $\hat{i}$, $\hat{j}$ and $\hat{k}$ are constant unit vectors.

Gradient of a Scalar Field:→

Consider a continious scalar point function ϕ. Let 1 and 2 are very closely spaced two level surfaces characterised by the values ϕ and $\phi + d\phi$ respectively.

'A' is a point on level surface 1.
'C' is a point on level surface 2 with position vector $\vec{dr}$ with respect to A.
'B' is a point on the second level surface which is the closest point of level surface 2 to A.



Let $\vec{AB} = \vec{dl}$.

Rate of change of ϕ along $\vec{AC}$ is $\frac{d\phi}{dr}$.

Rate of change of ϕ along $\vec{AB}$ is $\frac{d\phi}{dl}$

Since $dl \leq dr \Rightarrow \frac{d\phi}{dl} \geq \frac{d\phi}{dr}$

This shows that the rate of change of scalar point function ϕ is maximum in the direction $\vec{AB}$.

Let $\hat{n} \rightarrow$ a unit vector along $\vec{AB}$.

Now let us construct a vector "$\frac{d\phi}{dl}\hat{n}$". This is a vector point function (or field) whose magnitude at A is equal to the maximum rate of change of ϕ at A and whose direction is in the direction of maximum rate of change of ϕ at A.
This vector point function is called as "gradient of ϕ" at point A and is written as "grad ϕ".

So $\boxed{\text{grad}\,\phi = \frac{d\phi}{dl}\hat{n}}$ ——— (1)

Then $\text{grad}\,\phi \cdot \vec{dr} = \frac{d\phi}{dl}(\hat{n}\cdot\vec{dr}) = \frac{d\phi}{dl} \cdot dr\cos\theta$

Since $dr\cos\theta = dl$

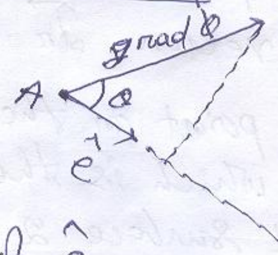
So $\boxed{\text{grad}\,\phi \cdot \vec{dr} = d\phi}$ ——— (2)

Hence the change in ϕ (i.e. $d\phi$) in displacing infinitesimally (by $\vec{dr}$) at A is equal to dot product of gradient of ϕ at A with $\vec{dr}$.

Equation (2) can be considered as the defining equation for "grad ϕ" at any point. Since eqn. (2) derivation is not done with reference to a specific co-ordinate system, hence this

equation (2') is valid in all co-ordinate systems.

Directional derivative of a function ϕ: →



The component of grad ϕ (at A) in the direction of a unit vector $\hat{e}$ is called as the directional derivative of ϕ in the direction of $\hat{e}$.

So
$$\left.\frac{d\phi}{dr}\right|_{\text{along } \hat{e}} = \hat{e}\cdot(\text{grad }\phi) \quad \text{--- (3')}$$

<u>Note:</u> → The above discussion shows that "gradient" is an operator which when operates on a scalar point function transforms it into a vector point function. Hence "grad" is considered as a vector operator and is represented by symbol "$\vec{\nabla}$" (called as "del"). So the symbolic representation of "grad ϕ" stands as "$\vec{\nabla}\phi$".

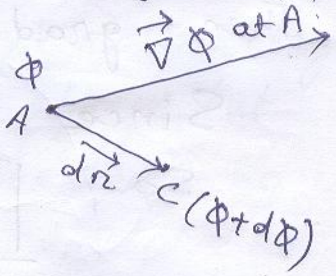
Expression for "grad ϕ" in Cartesian System: →

We know, in cartesian system: –

$$\phi = \phi(x,y,z)$$

⇒ Total differential "$d\phi$" $= \frac{\partial\phi}{\partial x}.dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}.dz$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$



But $\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz = \vec{dr}$

($\vec{dr}$ is the infinitesimal displacement from 'A' that results change in ϕ by $d\phi$.)

So $$d\phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\vec{dr} \quad \text{--- (4)}$$

But from eqn. (2') → $d\phi = \text{grad }\phi\cdot\vec{dr}$ --- (2')

Comparing equations (2') & (4), the expression for "grad ϕ" in Cartesian system is: →

$$\text{Grad}\, \phi = \vec{\nabla} \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \quad \text{---} (5)$$

Equation (5) Can also be written as: →

$$\vec{\nabla} \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \phi$$

Hence we can denote the del operator $\vec{\nabla}$ in Cartesian system as: →

$$\vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

Mark that here $\vec{\nabla}$ appears as a symbol which is not a quantity itself. But it is an operator which when operates on a scalar point function at any point, converts it to a vector point function, whose magnitude at that point is equal to the maximum rate of change of ϕ at that point, and whose direction is in the direction of maximum rate of change at that point. It is called as a vector operator, since mostly it behaves like a vector.

Example: → If $\phi = 4xy^2 + 2yz^2$, then find "grad ϕ" at point $(1, 2, 3)$.

Ans: → $\phi = 4xy^2 + 2yz^2 \Rightarrow \dfrac{\partial \phi}{\partial x} = \dfrac{\partial}{\partial x}(4xy^2 + 2yz^2)$

$$= 4y^2$$

$$\frac{\partial \phi}{\partial y} = 8xy + 2z^2, \quad \frac{\partial \phi}{\partial z} = 4yz$$

So "grad ϕ" at point (x, y, z) is: →

$$\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} = 4y^2 \hat{i} + (8xy + 2z^2)\hat{j} + 4yz\hat{k}$$

$$\Rightarrow \text{Grad}\, \phi \Big|_{(1,2,3)} = 4(2)^2 \hat{i} + \left[8 \cdot 1 \cdot 2 + 2 \cdot (3)^2\right] \hat{j} + 4 \cdot 2 \cdot 3\, \hat{k}$$

$$= 16\hat{i} + 34\hat{j} + 24\hat{k} \quad \underline{|\text{Ans.}}$$

Example: → Show that $\vec{\nabla} r = \hat{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$

Ans: → $r = (x^2 + y^2 + z^2)^{1/2} \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \dfrac{x}{r}$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

⑧

So $\vec{\nabla} r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}$

$$= \frac{\hat{i}x + \hat{j}y + \hat{k}z}{r} = \frac{\vec{r}}{r} = \hat{r}$$

Example: → Show that $\vec{\nabla} f(r) = \frac{df}{dr}\hat{r}$

Ans: → $\vec{\nabla} f(r) = \hat{i}\frac{\partial f(r)}{\partial x} + \hat{j}\frac{\partial f(r)}{\partial y} + \hat{k}\frac{\partial f(r)}{\partial z}$

$$= \hat{i}\frac{\partial f(r)}{\partial r}\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial f(r)}{\partial r}\cdot\frac{\partial r}{\partial y} + \hat{k}\frac{\partial f(r)}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{\partial f(r)}{\partial r}\cdot\frac{x}{r} + \hat{j}\frac{\partial f(r)}{\partial r}\cdot\frac{y}{r} + \hat{k}\frac{\partial f(r)}{\partial r}\cdot\frac{z}{r}$$

$$= \frac{df}{dr}\left[\frac{\hat{i}x + \hat{j}y + \hat{k}z}{r}\right] = \frac{df}{dr}\hat{r}$$

Task: → (i) Find $\vec{\nabla}\left(\frac{1}{r}\right)$, (ii) Find $\vec{\nabla}(\vec{a}\cdot\vec{r})$ where $\vec{a}$ is a const. vector

(iii) If $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$, $\vec{r}' = \hat{i}x' + \hat{j}y' + \hat{k}z'$

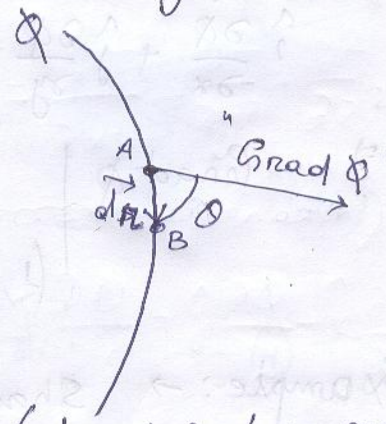
Find $\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$ and $\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

where $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ & $\vec{\nabla}' = \hat{i}\frac{\partial}{\partial x'} + \hat{j}\frac{\partial}{\partial y'} + \hat{k}\frac{\partial}{\partial z'}$,

(iv) Find $\vec{\nabla} f(|\vec{r}-\vec{r}'|)$ and $\vec{\nabla}' f(|\vec{r}-\vec{r}'|)$.

Theorem: → Gradient of ϕ at any point A is perpendicular to the level surface ϕ = const. passing through that point.

Proof: → If we displace by small amount $\vec{dr}$ ($= \vec{AB}$) on the level surface (ϕ = const) then from eqn ②, the change in ϕ is given by: →



$$d\phi = \vec{\nabla}\phi\cdot\vec{dr}.$$

Since A and B are on the same level surface, (Level surface passing through point B).

$$\phi\big|_{at\ A} = \phi\big|_{at\ B} \Rightarrow d\phi = 0$$

$\Rightarrow \quad \vec{\nabla}\phi \cdot d\vec{r} = 0 \quad \Rightarrow |\vec{\nabla}\phi|\, dr \cos\theta = 0$

Since $|\vec{\nabla}\phi| \neq 0$, $dr \neq 0$ So $\cos\theta = 0 \Rightarrow \theta = 90°$

Hence grad ϕ at A is perpendicular to the level surface passing through point A.

Note: → The above theorem can be used to obtain the direction of unit vector $\hat{n}$ at any point which is perpendicular to the level surface passing through that point. This $\hat{n}$ is also a unit vector drawn in the direction of maximum rate of change of ϕ at A.

$$\text{So} \quad \hat{n} = \frac{\text{Grad}\,\phi}{|\text{Grad}\,\phi|} \quad \text{——— (6)}$$

Example: → Find the unit vector normal to the surface $x^2+y^2+z^2=4$ at point $(0,2,0)$.

Ans: → Here $\phi = x^2+y^2+z^2=4 \Rightarrow \frac{\partial\phi}{\partial x} = 2x, \frac{\partial\phi}{\partial y} = 2y, \frac{\partial\phi}{\partial z} = 2z$

$$\Rightarrow \quad \vec{\nabla}\phi = 2(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\Rightarrow \quad \vec{\nabla}\phi\Big|_{at(0,2,0)} = 4\hat{j}$$

$\Rightarrow$ Unit vector perpendicular to $x^2+y^2+z^2=4$ at $(0,2,0)$ is: →

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{4\hat{j}}{4} = \hat{j}$$

Divergence of a vector Field: →

The vector operator $\vec{\nabla}$, though is an operator, yet can be operated on a vector point function $\vec{A}$ in two possible ways (dot and cross) giving meaningful quantities.

The operation of $\vec{\nabla}$ on $\vec{A}$ in a dot product way is called as "Divergence of $\vec{A}$" and is symbolised as: →

$$\text{Divergence of } \vec{A} = \vec{\nabla}\cdot\vec{A}.$$

Expression of $\vec{\nabla}\cdot\vec{A}$ in Cartesian System: →

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}A_x+\hat{j}A_y+\hat{k}A_z\right)$$

$$= \hat{i}\cdot\frac{\partial}{\partial x}\left(\hat{i}A_x+\hat{j}A_y+\hat{k}A_z\right)+\hat{j}\cdot\frac{\partial}{\partial y}\left(\hat{i}A_x+\hat{j}A_y+\hat{k}A_z\right)+\hat{k}\cdot\frac{\partial}{\partial z}\left(\hat{i}A_x+\hat{j}A_y+\hat{k}A_z\right)$$

$$= \hat{i}\cdot\hat{i}\frac{\partial A_x}{\partial x}+\hat{i}\cdot\hat{j}\frac{\partial A_y}{\partial x}+\hat{i}\cdot\hat{k}\frac{\partial A_z}{\partial x}$$
$$+\hat{j}\cdot\hat{i}\frac{\partial A_x}{\partial y}+\hat{j}\cdot\hat{j}\frac{\partial A_y}{\partial y}+\hat{j}\cdot\hat{k}\frac{\partial A_z}{\partial y}$$
$$+\hat{k}\cdot\hat{i}\frac{\partial A_x}{\partial z}+\hat{k}\cdot\hat{j}\frac{\partial A_y}{\partial z}+\hat{k}\cdot\hat{k}\frac{\partial A_z}{\partial z}$$

$$= \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

$\Rightarrow$ Divergence of $\vec{A} = \vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$ —— (7)

<u>Example:</u> $\rightarrow$ Find $\vec{\nabla}\cdot\vec{r}$

Ans: $\rightarrow$ $\vec{r} = \hat{i}x+\hat{j}y+\hat{k}z$ (So $A_x = x$, $A_y = y$ and $A_z = z$)

Then $\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 1+1+1 = 3$

Some Useful Identities: $\rightarrow$

1 $\rightarrow$ $\vec{\nabla}(\phi\pm\psi) = \vec{\nabla}\phi\pm\vec{\nabla}\psi$

2 $\rightarrow$ $\vec{\nabla}(\phi\psi) = \phi\vec{\nabla}\psi+\psi\vec{\nabla}\phi$

3 $\rightarrow$ $\vec{\nabla}\left(\frac{\phi}{\psi}\right) = \frac{\psi\vec{\nabla}\phi-\phi\vec{\nabla}\psi}{\psi^2}$

4 $\rightarrow$ $\vec{\nabla}\cdot(\phi\vec{A}) = \phi(\vec{\nabla}\cdot\vec{A})+\vec{A}\cdot(\vec{\nabla}\phi)$

5 $\rightarrow$ $\vec{\nabla}\cdot(\vec{A}\pm\vec{B}) = \vec{\nabla}\cdot\vec{A}\pm\vec{\nabla}\cdot\vec{B}$

6 $\rightarrow$ $\vec{\nabla}\cdot(\vec{\nabla}\phi) = \nabla^2\phi$

where ∇^2 is defined as $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

(This ∇^2 is called as Laplacian operator)

$$\vec{\nabla}\cdot\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Example: → Find $\nabla^2\phi$ when $\phi = x^2+y^2+z^2$

Ans: → $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

$= \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial y^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial z^2}(x^2+y^2+z^2)$

$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z) = 2+2+2 = 6.$

Solenoidal vector Field: → A vector field whose divergence is zero is called as a solenoidal vector field.

Curl of a vector Field: →

Curl of a vector field $\vec{A}$ is obtained by operating the vector operator $\vec{\nabla}$ on $\vec{A}$ in a cross product manner.

So Curl of $\vec{A} = \vec{\nabla} \times \vec{A}$

In cartesian system: →

$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ and $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

So $\vec{\nabla}\times\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left\{\hat{i}A_x(x,y,z) + \hat{j}A_y(x,y,z) + \hat{k}A_z(x,y,z)\right\}$

$$= \hat{i}\times\hat{i}\frac{\partial A_x}{\partial x} + \hat{i}\times\hat{j}\frac{\partial A_y}{\partial x} + \hat{i}\times\hat{k}\frac{\partial A_z}{\partial x}$$
$$+ \hat{j}\times\hat{i}\frac{\partial A_x}{\partial y} + \hat{j}\times\hat{j}\frac{\partial A_y}{\partial y} + \hat{j}\times\hat{k}\frac{\partial A_z}{\partial y}$$
$$+ \hat{k}\times\hat{i}\frac{\partial A_x}{\partial z} + \hat{k}\times\hat{j}\frac{\partial A_y}{\partial z} + \hat{k}\times\hat{k}\frac{\partial A_z}{\partial z}$$

$$= \hat{k}\frac{\partial A_y}{\partial x} + (-\hat{j})\frac{\partial A_z}{\partial x} + (-\hat{k})\frac{\partial A_x}{\partial y} + \hat{i}\frac{\partial A_z}{\partial y} + \hat{j}\frac{\partial A_x}{\partial z} + (-\hat{i})\frac{\partial A_y}{\partial z}$$

$$= \hat{i}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \hat{j}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \hat{k}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Irrotational or Lamellar vector field: → A vector field whose curl is zero is said to be "Lamellar" or "Irrotational" vector field.

Example: → Find "curl $\vec{r}$".

Ans: → $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

So $$\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0$$

So $\vec{r}$ is an irrotational vector field.

Some More useful identities: →

7 → $\vec{\nabla}\times(\vec{A}\pm\vec{B}) = \vec{\nabla}\times\vec{A} \pm \vec{\nabla}\times\vec{B}$

8 → $\vec{\nabla}\times(\phi\vec{A}) = \phi(\vec{\nabla}\times\vec{A}) + (\vec{\nabla}\phi)\times\vec{A}$

Example: → Find $\vec{\nabla}\times\{f(r)\hat{r}\}$

Ans: → $\vec{\nabla}\times[f(r)\hat{r}] = \{\vec{\nabla}f(r)\}\times\hat{r} + f(r)(\vec{\nabla}\times\hat{r})$

(using identity 8)

But $\vec{\nabla}f(r) = \frac{df}{dr}\hat{r}$

and $\vec{\nabla}\times\hat{r} = 0$ (prove it)

So $\vec{\nabla}\times f(r)\hat{r} = \frac{df}{dr}\hat{r}\times\hat{r} + 0 = 0$

Hence $f(r)\hat{r}$ is irrotational.

Example: → If a scalar field ϕ satisfies the relation $\nabla^2\phi = 0$, show that $\vec{\nabla}\phi$ is solenoidal as well as irrotational.

Ans: → $\nabla^2\phi = 0 \Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\phi) = 0$

So $\vec{\nabla}\phi$ is solenoidal

Also $$\text{curl}(\text{grad}\,\phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] + \hat{j}\left[\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \hat{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$= 0 + 0 + 0 = 0$$

So $\vec{\nabla} \times (\vec{\nabla} \phi) = 0$ for any scalar point function.

So $\vec{\nabla} \phi$ is irrotational.

Some More useful Identities :→

9 → $\text{Div}(\text{curl } \vec{A}) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

proof :→
$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \frac{\partial}{\partial x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

Since A_x, A_y and A_z are perfect differentials, hence

$$\frac{\partial^2 A_z}{\partial x \partial y} = \frac{\partial^2 A_z}{\partial y \partial x} \quad \text{and so on.}$$

So
$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

10 → $\text{Curl}(\text{grad } \phi) = \vec{\nabla} \times (\vec{\nabla} \phi) = 0$

11 → $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

12 → $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

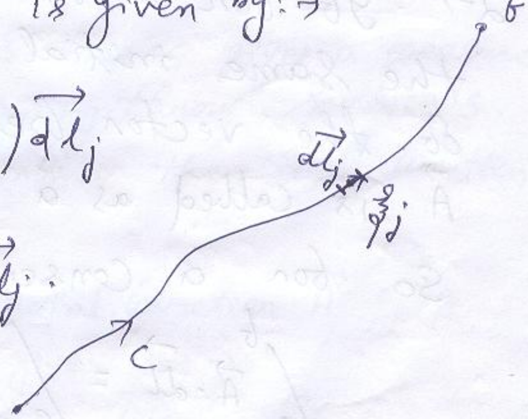
13 → $\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B}$

14 → $\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A} + (\vec{A} \cdot \vec{\nabla})\vec{B}$.

Integration of point functions :→

A → Line Integral :→

The line integral of a scalar point function ϕ from point i to point f along path C is given by :→



$$\int_{C\,i}^{f} \phi \, d\vec{l} = \lim_{n \to \infty} \sum_{j=1}^{n} \phi_j(q_j) \, d\vec{l}_j$$

where q_j is a point within segment $d\vec{l}_j$.

Since $\vec{dl} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

So
$$\int_{C\,i}^{f} \phi \, d\vec{l} = \hat{i}\int_{C\,i}^{f} \phi \, dx + \hat{j}\int_{C\,i}^{f} \phi \, dy + \hat{k}\int_{C\,i}^{f} \phi \, dz$$

Similarly, the line integral of a vector field $\vec{A}$ from point i to point f along path c in a dot product way is:→

$$\int_{c\,i}^{f} \vec{A}\cdot\vec{dl} = \underset{n\to\infty}{\text{limit}} \sum_{j=1}^{n} \vec{A}_j(\xi_j)\cdot\vec{dl}_j$$

where $\vec{A}_j(\xi_j)$ is the value of $\vec{A}_j$ at a point ξ_j taken within segment $\vec{dl}_j$.

The line integral of a vector field $\vec{A}$ from point i to point f along path c in a cross product way is:→

$$\int_{c\,i}^{f} \vec{A}\times\vec{dl} = \underset{n\to\infty}{\text{limit}} \sum_{j=1}^{n} \vec{A}_j(\xi_j)\times\vec{dl}_j$$

We can write $\int_{c\,1}^{2} \vec{A}\cdot\vec{dl}$ as :→

$$\int_{c\,1}^{2} \vec{A}\cdot\vec{dl} = \int_{c\,1}^{2} A_x\,dx + \int_{c\,1}^{2} A_y\,dy + \int_{c\,1}^{2} A_z\,dz$$

and

$$\int_{c\,1}^{2} \vec{A}\times\vec{dl} = \hat{i}\int_{c\,1}^{2}(A_y\,dz - A_z\,dy) + \hat{j}\int_{c\,1}^{2}(A_z\,dx - A_x\,dz) + \hat{k}\int_{c\,1}^{2}(A_x\,dy - A_y\,dx)$$

Note:→

1→ These line integrals generally depend on the path 'c' followed i.e. keeping points i and f same, if we change the path of integration, then the line integral changes.

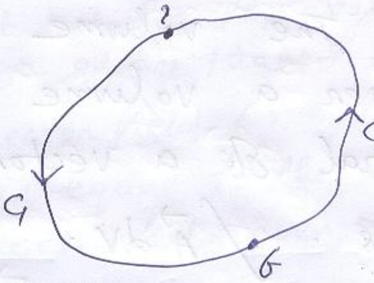
2→ If for different paths c_1, c_2, c_3 --- etc. connecting the same initial and final points, the line integral of the vector field $\vec{A}$ given by $\int_1^2 \vec{A}\cdot\vec{dl}$ is same, then $\vec{A}$ is called as a conservative vector field.

So for a conservative vector field:→

$$\int_{c_1\,i}^{f} \vec{A}\cdot\vec{dl} = \int_{c_2\,i}^{f} \vec{A}\cdot\vec{dl} = \int_{c_3\,i}^{f} \vec{A}\cdot\vec{dl} = ---- \text{ \& so on.}$$

3→ For a conservative vector field $\oint \vec{A}\cdot\vec{dl} = 0$, where $\oint$ stands for integration over a closed path.

proof:→ Let a closed path C consists of two paths C_1 & C_2 as shown in figure



Then $$\int_{C_1,\, i}^{f} \vec{A}\cdot\vec{dl} = \int_{C_2,\, i}^{f} \vec{A}\cdot\vec{dl}$$
(If $\vec{A}$ is conservative field).

Again $$\int_{C_2,\, i}^{f} \vec{A}\cdot\vec{dl} = -\int_{C_2,\, f}^{i} \vec{A}\cdot\vec{dl}$$

So $$\int_{C_1,\, i}^{f} \vec{A}\cdot\vec{dl} = -\int_{C_2,\, f}^{i} \vec{A}\cdot\vec{dl} \Rightarrow \int_{C_1,\, i}^{f} \vec{A}\cdot\vec{dl} + \int_{C_2,\, f}^{i} \vec{A}\cdot\vec{dl} = 0$$

$$\Rightarrow \oint_{C_1+C_2} \vec{A}\cdot\vec{dl} = 0 \Rightarrow \oint_{C} \vec{A}\cdot\vec{dl} = 0$$

B→ <u>Surface Integral:→</u>

(i) The surface integral of a scalar point function ϕ over a surface S is defined as:→
$$\int_S \phi\, \vec{ds} = \lim_{n\to\infty} \sum_{j=1}^{n} \phi(q_j)\cdot \vec{ds_j}$$
where $\phi(q_j)$ is the value of ϕ at a point q_j (where q_j is a point on the elementary surface ds_j on S).

(ii) The surface integral of a vector point function $\vec{A}$ over a surface S in a dot product way is:→
$$\int_S \vec{A}\cdot\vec{ds} = \lim_{n\to\infty} \sum_{j=1}^{n} \vec{A}(q_j)\cdot \vec{ds_j}$$

This surface integral of $\vec{A}$ over surface S is called as "flux of vector field $\vec{A}$" over S.

This flux of vector field $\vec{A}$ over surface S gives a measure of the number of lines of forces passing through surface S.

If the surface S is a closed surface, then it is written as $\oint \vec{A}\cdot\vec{ds}$.

(iii) The surface integral of a vector point function $\vec{A}$ over surface S in a cross product way is given as:→
$$\int_S \vec{A}\times\vec{ds} = \lim_{n\to\infty} \sum_{j=1}^{n} \vec{A}(q_j)\times \vec{ds_j}$$

C → Volume Integral :→

The volume integral of a scalar point function ϕ over a volume V is given as "$\int_V \phi\, dV$" and the volume integral of a vector point function $\vec{A}$ over the volume V is $\int_V \vec{A}\, dV$.

Definitions of Gradient, Divergence and Curl in terms of Integrals :→

1. Gradient :→ The gradient of a scalar field ϕ at any point P is the surface integral of the scalar point function ϕ over a unit volume, calculated for an infinitesimal volume element 'ΔV' drawn centred about P and bound by an infinitesimal surface element ΔS.

i.e Gradient of ϕ at P $= \lim\limits_{\Delta V \to 0} \dfrac{\oint\limits_{\Delta S} \phi\, \vec{ds}}{\Delta V}$

2. Divergence :→ The divergence of a vector point function $\vec{A}$ at a point P is equal to the surface integral of $\vec{A}$ per unit volume calculated over an infinitesimal volume element ΔV drawn centred about P and bound by surface S.

$$\text{So}\quad \text{Div}\, \vec{A} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta V}$$

This can also be defined as the outward flux per unit volume drawn centred about P.

3. Curl :→ The curl of a vector point function $\vec{A}$ at a point P is the maximum line integral of $\vec{A}$ per unit area, calculated over a closed path C drawn centred about point A and bounding surface ΔS.

$$\text{i.e}\quad \text{Curl}\, \vec{A} = \lim_{\Delta S \to 0} \frac{\oint \vec{A} \cdot \vec{dl}}{\Delta S}\, \hat{n}$$

The direction of this curl is along the unit normal $\hat{n}$ drawn to area ΔS when it is in the orientation for which the line integral along the path bounding it is maximum.

Gauss's divergence theorem: →

This is an integral theorem that relates the volume integral with the surface integral. It states that: →

" The surface integral of a vector field (i.e flux of the vector field) over a closed surface is equal to the volume integral of the divergence of the vector field over the volume that is bound by the closed surface".

Mathematically, $\oint_S \vec{A} \cdot \vec{ds} = \int_V (\vec{\nabla} \cdot \vec{A}) dV$

where V is the volume bound by the closed surface S.

Example: → Find $\oint_S \vec{r} \cdot \vec{ds}$.

Ans: → Using Gauss's divergence theorem,

$$\oint_S \vec{r} \cdot \vec{ds} = \int_V (\vec{\nabla} \cdot \vec{r}) dV = \int_V 3\, dV = 3 \int_V dV = 3V$$

Stoke's Theorem: →

This theorem relates the surface integral of a vector field with the line integral. It states that,

" The line integral of a vector field over a closed curve C is equal to the surface integral of the curl of vector field over the surface bound by the curve C".

Mathematically, $\oint_C \vec{A} \cdot \vec{dl} = \int_S (\vec{\nabla} \times \vec{A}) \cdot \vec{ds}$

Example: → Find $\oint_C \vec{r} \cdot \vec{dl}$

Ans: → $\oint_C \vec{r} \cdot \vec{dl} = \int_S (\vec{\nabla} \times \vec{r}) \cdot \vec{ds} = 0$ (since $\vec{\nabla} \times \vec{r} = 0$)

Example: → Show that electric force is a conservative force.

Ans: → The electric force can be written in the form,

$$\vec{F} = f(r)\hat{r} \quad \text{where} \quad f(r) = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}$$

Then work done by the electric force over a closed path is: →

$$W = \oint_C \vec{F} \cdot \vec{dl} = \oint_C f(r)\hat{r} \cdot \vec{dl} = \int_S [\vec{\nabla} \times f(r)\hat{r}] \cdot \vec{ds}$$
$$= 0.$$

Hence electric force is a conservative vector field

Theorem:→ Show that for a conservative vector field, it's curl vanishes.

Ans:→ If $\vec{A}$ is a conservative vector field then:→

$$\oint \vec{A}\cdot d\vec{l} = 0$$

Using stoke's theorem, $\oint \vec{A}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{A}\cdot d\vec{s} = 0$

$$\Rightarrow \vec{\nabla}\times\vec{A} = 0.$$

Note:→ we have seen previously $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$.

If $\vec{A}$ is conservative, $\vec{\nabla}\times\vec{A} = 0$

comparing the two, '$\vec{A}$' can be written as gradient of a scalar point function as:→

$$\vec{A} = \vec{\nabla}\phi.$$

So every conservative vector field can be written as gradient of a scalar point function ϕ.

Green's Theorem:→

The green's theorem or Green's identity states that, " For two continious scalar point functions ϕ and ψ:

$$\oint_S (\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi)\cdot d\vec{s} = \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV$$

This theorem can be obtained as follows.

For any scalar field ϕ and a vector field $\vec{A}$, we have:→

$$\vec{\nabla}\cdot(\phi\vec{A}) = (\vec{\nabla}\phi)\cdot\vec{A} + \phi\,\vec{\nabla}\cdot\vec{A}$$

writing $\vec{A} = \vec{\nabla}\psi$, where ψ is a scalar field

$$\vec{\nabla}\cdot(\phi\vec{\nabla}\psi) = (\vec{\nabla}\phi)\cdot(\vec{\nabla}\psi) + \phi\,\vec{\nabla}\cdot(\vec{\nabla}\psi)$$

$$\Rightarrow \vec{\nabla}\cdot(\phi\vec{\nabla}\psi) = (\vec{\nabla}\phi)\cdot(\vec{\nabla}\psi) + \phi\nabla^2\psi$$

Simillarly interchanging ϕ and ψ,

$$\vec{\nabla}\cdot(\psi\vec{\nabla}\phi) = (\vec{\nabla}\psi)\cdot(\vec{\nabla}\phi) + \psi\nabla^2\phi$$

Then $\vec{\nabla}\cdot\left[\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right] = \phi\nabla^2\psi - \psi\nabla^2\phi$

Integrating both sides over a volume V,

$$\int_V \vec{\nabla}\cdot\left[\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right]dv = \int_V\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dv$$

The L.H.S can be transformed into a surface integral by using Gauss's divergence theorem as:→

$$\oint_S\left(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right)\cdot\vec{ds} = \int_V\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dv \quad \text{(proved)}$$

Some More Examples:→

1 → If vectors $\vec{A}$ and $\vec{B}$ are irrotational, show that $\vec{A}\times\vec{B}$ is solenoidal.

Ans:→ Since $\vec{A}$ and $\vec{B}$ are irrotational, $\vec{\nabla}\times\vec{A}=0$ and $\vec{\nabla}\times\vec{B}=0$

But $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) = 0$

So $\vec{A}\times\vec{B}$ is solenoidal.

2 → Show that $\int_S(\vec{\nabla}\times\vec{r})\cdot\vec{ds} = \oint_C(x\,dx + y\,dy + z\,dz)$

Ans:→ $\int(\vec{\nabla}\times\vec{r})\cdot\vec{ds} = \oint_C \vec{r}\cdot\vec{dl}$ (using stoke's theorem)

$$= \oint_C(x\,dx + y\,dy + z\,dz)$$

3 → Evaluate $\vec{\nabla}\left(-\frac{K}{r}\right)$ where K is a constant.

Ans:→
$$\vec{\nabla}\left(-\frac{K}{r}\right) = -K\,\vec{\nabla}\left(\frac{1}{r}\right) = -K\left[\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]$$
$$= -K\left[\hat{i}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x} + \hat{j}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial y} + \hat{k}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\cdot\frac{\partial r}{\partial z}\right]$$
$$= -K\left[\hat{i}\left(-\frac{1}{r^2}\right)\cdot\frac{x}{r} + \hat{j}\left(-\frac{1}{r^2}\right)\frac{y}{r} + \hat{k}\left(-\frac{1}{r^2}\right)\frac{z}{r}\right]$$
$$= +\frac{K}{r^2}\left[\frac{\hat{i}x + \hat{j}y + \hat{k}z}{r}\right] = \frac{K}{r^2}\hat{r}$$

Task:→

1 → Evaluate $\vec{\nabla} r^n$.

2 → Evaluate $\vec{\nabla}\cdot\vec{a}$ and $\vec{\nabla}\times\vec{a}$ {where $\vec{a}$ = const. vector}

3 → Evaluate $\vec{\nabla}\cdot(r^n\hat{r})$ and find value of n for which this divergence vanishes.

4 → If f and g are two scalar fields, show that $(\vec{\nabla}f\times\vec{\nabla}g)=0$

5 → Show that $\vec{\nabla}\cdot(\vec{a}\times\vec{r})=0$ where $\vec{a}$ is a constant vector.